Velocity of Longitudinal wave in a Solid

Consider a rod of uniform area of cross-section a.

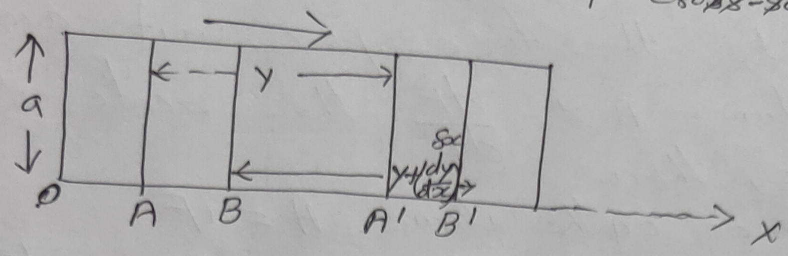

fig ①

Displacement of plane B = BB'

$$= y+\left(\frac{dy}{dx}\right)\delta x$$

As the displacement of the plane B is greater than the displacement of the plane A, the distance B'A' is greater than BA by an amount given by

$$y+\left(\frac{dy}{dx}\right)\delta x - y = \left(\frac{dy}{dx}\right)\delta x$$

$\therefore$ change of distance betn the planes A & B

$$= \left(\frac{dy}{dx}\right)\delta x$$

and original distance betn the planes A & B $= \delta x$.

$$\therefore \text{Longitudinal strain} = \frac{\text{Change in length}}{\text{original length}}$$

$$= \frac{\text{change of distance betn the planes A\&B}}{\text{original distance betn the plane A\&B}}$$

$$= \frac{\left(\frac{dy}{dx}\right)\delta x}{\delta x}$$

$$\Rightarrow \text{Longitudinal strain} = \frac{dy}{dx}$$

When the rod is under strain elastic forces automatically come into play.

Let F be the force per unit area on the plane A & $\frac{dF}{dx}$ the rate of change of force, then

Force on the plane A $= aF$

& Force on the plane B $= aF + a\left(\frac{dF}{dx}\right)\delta x$.

Hence net force on the layer A,B

$$= a\left(\frac{dF}{dx}\right)\delta x$$

Mass of the medium in the layer AB

$$= a\,\delta x\,\rho.$$

where ρ = density of the medium.

If $\frac{d^2y}{dt^2}$ represents the acceleration of the particles in the layer, then

$$a\left(\frac{dF}{dx}\right)\delta x = a\,\delta x\,\rho\frac{d^2y}{dt^2}$$

or $$\frac{dF}{dx} = \rho\frac{d^2y}{dt^2} \quad \text{——— (1)}$$

$$\because Y = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{\frac{dy}{dx}}$$

$$\therefore \quad F = Y\cdot\frac{dy}{dx}$$

or $$\frac{dF}{dx} = \frac{d}{dx}\left(Y\frac{dy}{dx}\right) = Y\frac{d^2y}{dx^2} \quad \text{——— (2)}$$

Comparing equⁿ ① & ② we have

$$\rho \frac{d^2y}{dt^2} = Y \frac{d^2y}{dx^2}$$

or, $$\frac{d^2y}{dt^2} = \frac{Y}{\rho} \frac{d^2y}{dx^2}$$

This expression represents the differential equⁿ of longitudinal wave propagation along the rod. Comparing it with standard equⁿ of wave motion.

$$\frac{d^2y}{dt^2} = v^2 \frac{d^2y}{dx^2}$$

we get the velocity of the wave

$$v^2 = \frac{Y}{\rho}$$

or, $$v = \sqrt{\frac{Y}{\rho}}$$

This shows that the velocity of longitudinal waves travelling through a rod depends upon young's modulus Y and density ρ. It is independent of the applied force and area of cross-section.

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